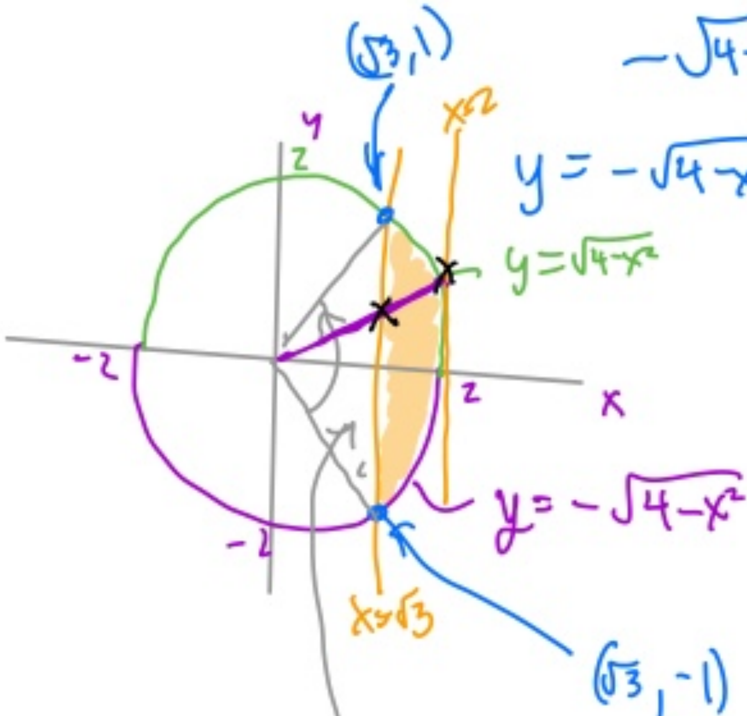
Warmups: (1) $$\int_{\sqrt{3}}^{2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} 2x^2 y \, dy \, dx.$$

Convert to polar coordinates.

$\sqrt{3} \le x \le 2$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$y = -\sqrt{4-x^2}, \quad y = \sqrt{4-x^2}$



$y^2 = 4 - x^2$

$x^2 + y^2 = 4$

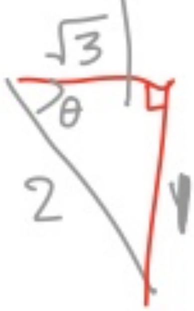
$x = \sqrt{3}$

$r\cos\theta = \sqrt{3}$

$r = \frac{\sqrt{3}}{\cos\theta}$

$\frac{\sqrt{3}}{\cos\theta} \le r \le 2$

$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$



$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

integral $= \boxed{\int_{\theta=-\pi/6}^{\pi/6} \int_{r=\sqrt{3}/\cos\theta}^{2} 2r^2\cos^2\theta \cdot r\sin\theta \; r\,dr\,d\theta}$

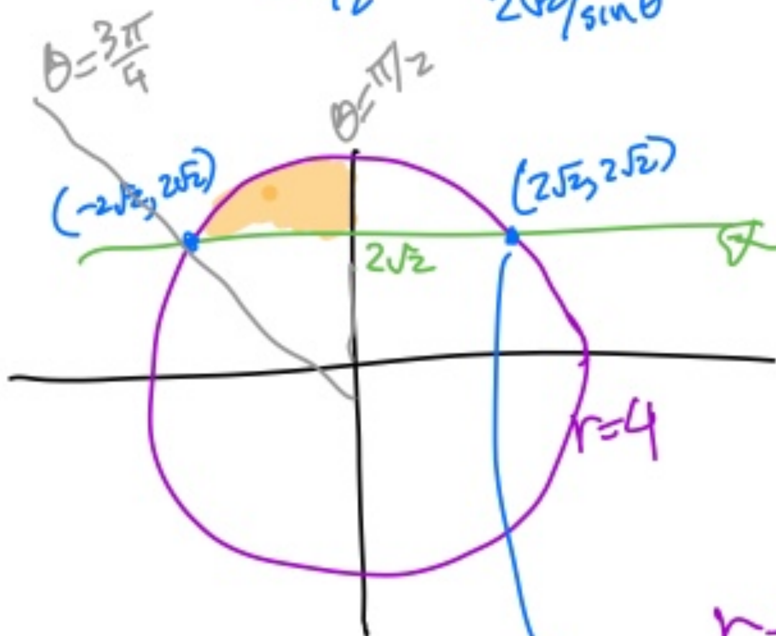
② $$\int_{\pi/2}^{3\pi/4}\int_{2\sqrt{2}/\sin\theta}^{4}\int_{0}^{r^5\sin\theta}\frac{\cos\theta}{r^2}\,dz\,dr\,d\theta$$

ⓐ Do the z part of the integral.

ⓑ Draw the region of integration in the xy plane.

ⓐ $$\int_{\pi/2}^{3\pi/4}\int_{2\sqrt{2}/\sin\theta}^{4}\left(\frac{\cos\theta}{r^2}\,z\,\Big|_{0}^{r^5\sin\theta}\right)dr\,d\theta$$

$$=\int_{\pi/2}^{3\pi/4}\int_{2\sqrt{2}/\sin\theta}^{4}\frac{\cos\theta}{\cancel{r^2}}\left(\overset{r^3}{\cancel{r^5}}\sin\theta\right)dr\,d\theta$$



$\frac{2\sqrt{2}}{\sin\theta}\le r\le 4$

$\frac{\pi}{2}\le\theta\le\frac{3\pi}{4}$.

$r=\frac{2\sqrt{2}}{\sin\theta}$

$\Leftrightarrow\ r\sin\theta=2\sqrt{2}$

$y=2\sqrt{2}$

$r=4$

$y=2\sqrt{2}$

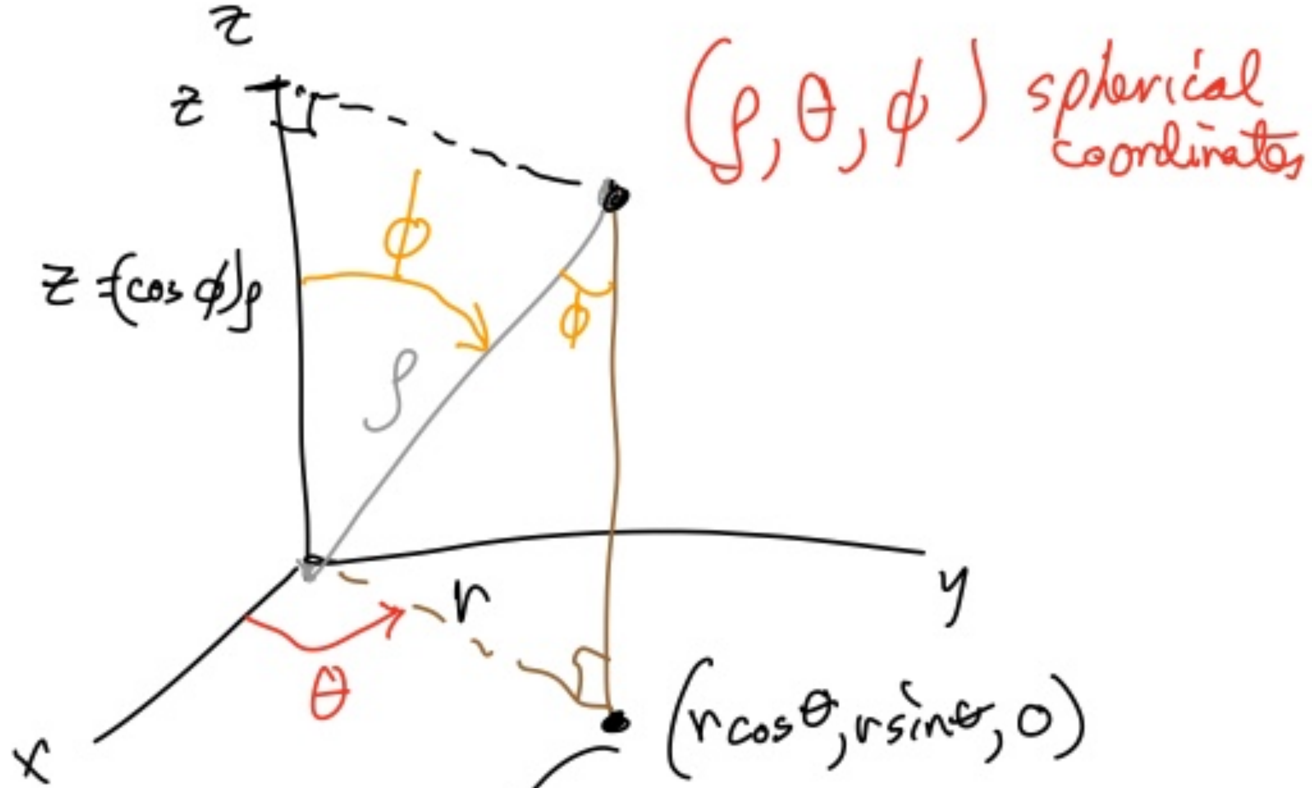
$x=$

$x^2+y^2=16\Rightarrow x^2+8=16$

$x^2=8$

$x=\pm2\sqrt{2}$

Spherical Coordinates



$r = \rho \cdot \sin\phi$

$= (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, 0)$

Spherical Coordinates.

$$(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$$

Question: what is $dx\,dy\,dz$ in the new coordinates?

<u>Method 1</u>: Use differential forms.

$$x = \rho \sin\phi \cos\theta$$

$$dx = \sin\phi \cos\theta\, d\rho - \rho \sin\phi \sin\theta\, d\theta + \rho \cos\phi \cos\theta\, d\phi$$

$$y = \rho \sin\phi \sin\theta$$

$dy = \ldots$

$z = \rho\cos\phi$

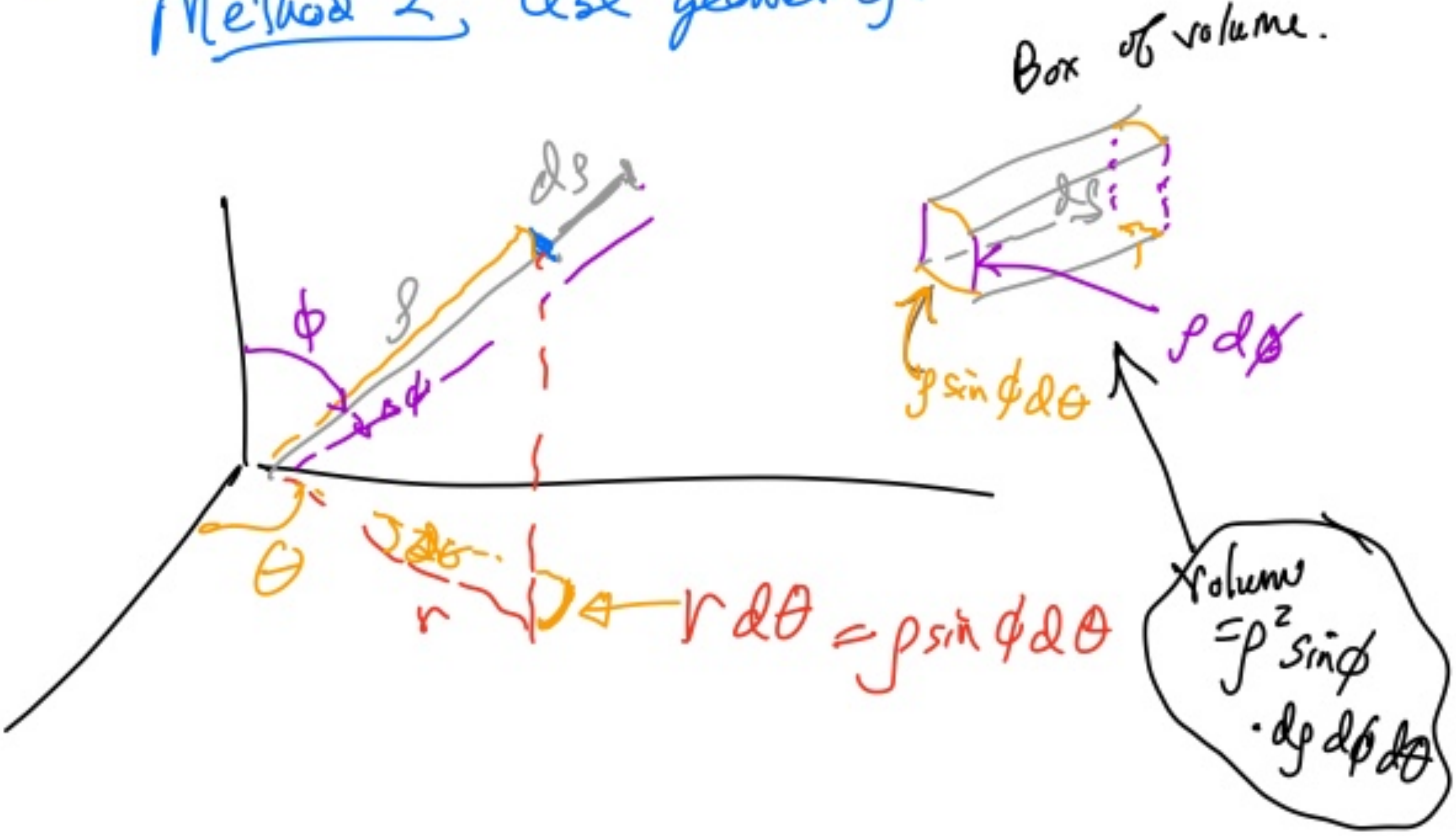
$dz = \cos\phi\, d\rho - \rho\sin\phi\, d\phi.$

$\Rightarrow \quad dx \wedge dy \wedge dz = \underline{\qquad\qquad}\, d\rho \wedge d\theta \wedge d\phi$

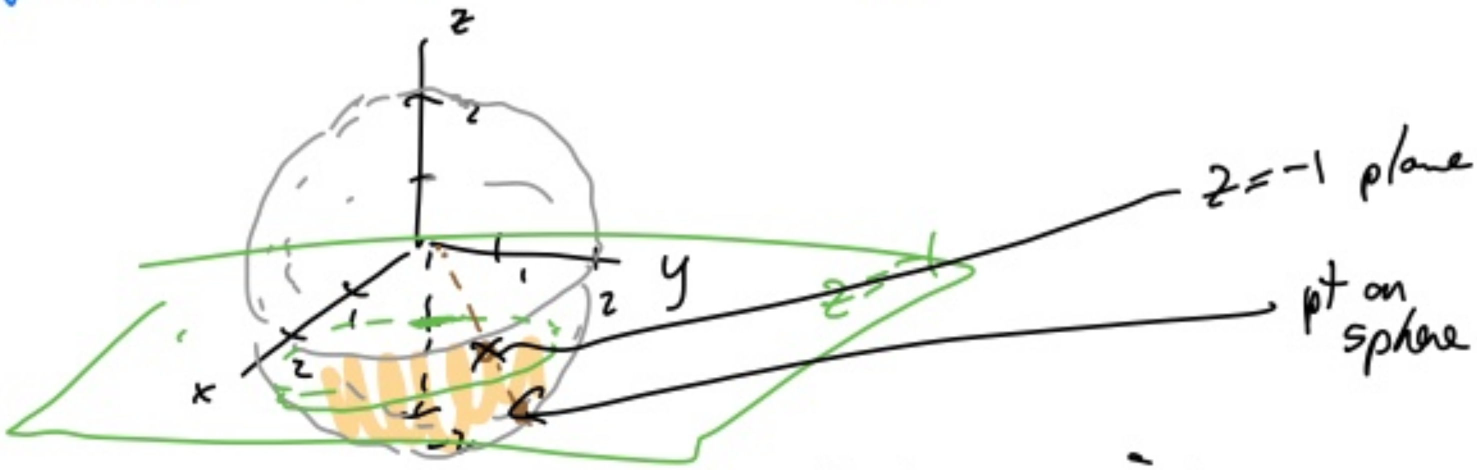
$$dx\,dy\,dz = \rho^2 \underline{\sin\phi}\; d\rho\, d\theta\, d\phi$$

Extra Credit: Do the calculation.

Method 2, Use geometry.



Back to question from last class:
Find the volume of the portion of the ball of radius 2 centered at the origin that has $z \le -1$

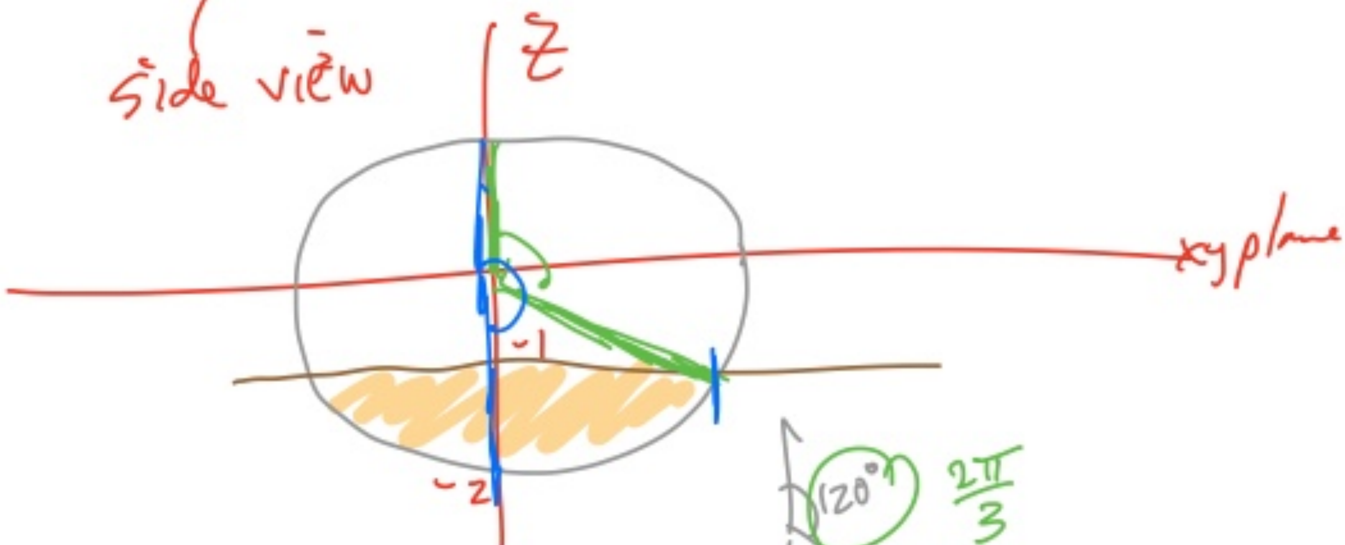


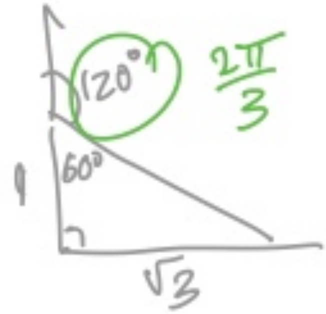
What is this region in spherical coordinates.

$$-\frac{1}{\cos\phi} \le \rho \le 2$$
$$0 \le \theta \le 2\pi$$
$$\frac{2\pi}{3} \le \phi \le \pi$$

$z = -1$
$\rho\cos\phi = -1$
$\rho = \frac{-1}{\cos\phi}$

side view





$$\Rightarrow \text{Volume} = \int_{\phi=\frac{2\pi}{3}}^{\phi=\pi} \int_{\theta=0}^{2\pi} \int_{\rho=\frac{-1}{\cos\phi}}^{2} \underline{1}\cdot\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_{\phi=\frac{2\pi}{3}}^{\pi} \int_{\theta=0}^{2\pi} \left(\frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=\frac{-1}{\cos\phi}}^{2} \right) d\theta \, d\phi$$

$$= \int_{\phi=\frac{2\pi}{3}}^{\pi} \int_{\theta=0}^{2\pi} \left(\frac{8}{3}\sin\phi + \frac{1}{3}\underbrace{\frac{\sin\phi}{(\cos^3\phi)}}_{\tan\phi\sec^2\phi} \right) d\theta \, d\phi$$

$$= \int_{\phi=\frac{2\pi}{3}}^{\pi} \frac{16\pi}{3}\sin\phi + \frac{2\pi}{3}\tan\phi\sec^2\phi \, d\phi$$

$$= -\frac{16\pi}{3}\cos\phi + \frac{2\pi}{3}\frac{\tan^2\phi}{2} \Big|_{2\pi/3}^{\pi}$$

$$= -\frac{16\pi}{3}(-1) + \frac{2\pi}{3}\cdot\frac{0}{2} - \left(-\frac{16\pi}{3}\left(\frac{-1}{2}\right) + \frac{2\pi}{3}\frac{(-\sqrt{3})^2}{2} \right)$$

$$= \frac{16\pi}{3} + 0 - \frac{8\pi}{3} - \pi = \frac{8\pi}{3} - \frac{3\pi}{3} = \boxed{\frac{5\pi}{3}}$$

$$= \boxed{\frac{5\pi}{3}}$$